On the two-dimensional Leznov lattice equation with self-consistent sources

2007 J. Phys. A: Math. Theor. 40 12691

(http://iopscience.iop.org/1751-8121/40/42/S13)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 12691–12700 doi:10.1088/1751-8113/40/42/S13

On the two-dimensional Leznov lattice equation with self-consistent sources

Hong-Yan Wang[1,2], Xing-Biao Hu[1] and Hon-Wah Tam[3]

[1] Institute of Computational Mathematics and Scientific Engineering Computing, AMSS, Chinese Academy of Sciences, PO Box 2719, Beijing 100080, People's Republic of China
[2] Graduate School of the Chinese Academy of Sciences, Beijing, People's Republic of China
[3] Department of Computer Science, Hong Kong Baptist University, Kowloon Tong, Hong Kong, People's Republic of China

E-mail: wanghy@lsec.cc.ac.cn, hxb@lsec.cc.ac.cn and tam@comp.hkbu.edu.hk

Received 30 October 2006, in final form 11 December 2006
Published 2 October 2007
Online at stacks.iop.org/JPhysA/40/12691

Abstract

A two-dimensional Leznov lattice equation with self-consistent sources (LeznovESCS) is presented through the 'source generation' procedure, starting from the Casoratian determinant solution of the two-dimensional Leznov lattice equation. As a result, the Casoratian determinant solution of the LeznovESCS is obtained. Besides, we also give the Grammian determinant solution of the coupled system. In addition, the bilinear Bäcklund transformation (BT) for the LeznovESCS is given.

PACS numbers: 02.30.Ik, 02.30.Ks, 02.30.Zz

1. Introduction

Soliton equations with self-consistent sources (SESCSs) exhibit various nonlinear dynamics, and they are extensively treated within the framework of constrained flows of soliton equations [1–5]. Some SESCSs have important physical applications as well, particularly to problems in hydrodynamics, plasma physics, solid-state physics, among others [6–8]. SESCSs have been studied and solved in several ways, such as the inverse scattering transform, Darboux transformation and Hirota's method (see [9–25]). Recently, Hu and Wang proposed a new algebraic procedure called 'source generation'[26] to construct and solve SESCSs, based on the bilinear forms of the original equations without sources.(In [26], this new procedure was called 'source generalization procedure' as first. While considering the fact that the procedure enables one to introduce sources in integrable equations, it may be more precise to rename the title of the procedure as 'source generation'.) Through the procedure we have obtained determinant or Pfaffian solutions of SESCSs which are closely related to solutions of the original equations without sources. In this case, bilinear SESCSs are nothing but Pfaffian identities. The procedure has been successfully applied to the 2D Toda lattice equation, BKP-type and discrete KP equations, and so on [26, 27]. There are three steps involved in the source generation procedure:

1. to express N-soliton solutions of a soliton equation without sources in the form of determinant or Pfaffian with some arbitrary constants, say $c_{i,j}$;
2. to introduce the corresponding determinant or Pfaffian with arbitrary functions of an independent variable, e.g. $c_{i,j}(t)$;
3. to seek coupled bilinear equations whose solutions are the above generalized determinants or Pfaffians. The coupled system is just the SESCS.

Obviously, the success of source generation procedure relies heavily on some arbitrary constants appearing in determinantal solutions or Pfaffian solutions of soliton equations without sources. Since Grammian determinant solutions or Grammian Pfaffian solutions of soliton equations without sources always contain arbitrary constants explicitly, say $c_{i,j}$, naturally we prefer to begin with Grammian determinant solutions or Grammian Pfaffian solutions. In [26, 27], we have constructed and solved four SESCSs starting from the Grammian determinant solution or Grammian Pfaffian solution of soliton equations without sources. Until now, all examples of SESCSs found by the source generation procedure always choose Grammian determinant or Grammian Pfaffian solutions as a starting point. However, as we know, most AKP-type soliton equations (such as the KP equation or the 2D Toda lattice equation) have not only Grammian determinant solutions, but also Wronskian or Casoratian determinant solutions. Hence it is natural to ask if we can also apply the source generation procedure to the Wronskian (or Casoratian) determinant solution of soliton equations. The answer is affirmative. Although different from Grammian determinant solutions, some arbitrary constants hidden in the forms of Wronskian or Casoratian determinant solutions can be utilized. The purpose of this paper is to apply the source generation procedure to the two-dimensional Leznov lattice equation, starting from the Casoratian determinant solution of the Leznov lattice equation.

The two-dimensional Leznov lattice equation is given by [28]

$$\frac{\partial^2}{\partial x\partial y}\ln\theta(n) = \theta(n+1)p(n+1) - 2\theta(n)p(n) + \theta(n-1)p(n-1), \tag{1}$$

$$\frac{\partial p(n)}{\partial y} = \theta(n+1) - \theta(n-1). \tag{2}$$

Introducing the auxiliary variable z and dependent variable transformation:

$$\theta(n) = \frac{\tau(n+1)\tau(n-1)}{\tau(n)^2}, \qquad p(n) = \frac{1}{2}\frac{D_xD_y\tau(n)\cdot\tau(n)}{\tau(n+1)\tau(n-1)},$$

the Leznov equation can be transformed into the bilinear equations [30]:

$$(D_yD_z - 2\,\mathrm{e}^{D_n} + 2)\tau(n)\cdot\tau(n) = 0, \tag{3}$$

$$(D_yD_x - 2D_z\,\mathrm{e}^{D_n})\tau(n)\cdot\tau(n) = 0, \tag{4}$$

where D is the Hirota bilinear operator [29]

$$D_x^mD_t^nf(x,t)\cdot g(x,t) = \frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial s^n}f(x+y,t+s)g(x-y,t-s)|_{s=0,y=0}, m,n = 0,1,\ldots$$

$$\exp(\delta D_n)f_n\cdot g_n \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right]f(n)g(n')|_{n=n'} = f(n+\delta)g(n-\delta).$$

Equations (3), (4) have the following Casoratian determinant solution [30, 31]:

$$\tau_n \triangleq \tau(n) = \begin{vmatrix} \varphi_1(n) & \varphi_1(n+1) & \cdots & \varphi_1(n+N-1) \\ \varphi_2(n) & \varphi_2(n+1) & \cdots & \varphi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N(n) & \varphi_N(n+1) & \cdots & \varphi_N(n+N-1) \end{vmatrix}, \tag{5}$$

where each function $\varphi_i(n)$ satisfies the relations

$$\frac{\partial \varphi_i(n)}{\partial y} = \varphi_i(n+1), \qquad \frac{\partial \varphi_i(n)}{\partial z} = -\varphi_i(n-1), \qquad \frac{\partial \varphi_i(n)}{\partial x} = -\varphi_i(n-2). \tag{6}$$

A particular solution of (6) can be obtained by choosing the following 'exponential type' function:

$$\varphi_i(n) = \alpha_i p_i^n \,\mathrm{e}^{-p_i^{-2}x + p_i y - \frac{1}{p_i} z} + \beta_i q_i^n \,\mathrm{e}^{-q_i^{-2}x + q_i y - \frac{1}{q_i} z}, \tag{7}$$

where p_i, q_i, α_i and β_i are arbitrary constants.

2. Construction of the Leznov lattice equation with self-consistent sources

In this section, we will apply the 'source generation' procedure to the bilinear Leznov lattice equations (3) and (4), starting from the Casoratian determinant solution of the Leznov lattice equation. Following the source generation procedure, we generalize τ_n in (5) into a new function f_n such that

$$f_n \triangleq f(n) = \begin{vmatrix} \psi_1(n) & \psi_1(n+1) & \cdots & \psi_1(n+N-1) \\ \psi_2(n) & \psi_2(n+1) & \cdots & \psi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(n) & \psi_N(n+1) & \cdots & \psi_N(n+N-1) \end{vmatrix}, \tag{8}$$

with each function $\psi_i(n)$ expressed as

$$\psi_i(n) = \varphi_{i1}(n) + (-1)^{i-1} C_i(x) \varphi_{i2}(n), \qquad i = 1, 2, \ldots, N, \tag{9}$$

where each function $\varphi_{i1}(n), \varphi_{i2}(n)$ satisfy the dispersion relation (6). Here $C_i(x)$ is defined by

$$C_i(x) = \begin{cases} \beta_i(x), & 1 \leqslant i \leqslant K \leqslant N,\ K, N \in Z^+, \\ \beta_i, & \text{otherwise}, \end{cases} \tag{10}$$

where $\beta_i(x)$ is an arbitrary function of the variable x.

For the sake of convenience in calculations, we now express f_n in (8) in Pfaffian form [29]:

$$f_n = \mathrm{pf}(d_0, d_1, \ldots, d_{N-1}, N, \ldots, 2, 1)_n, \tag{11}$$

where Pfaffian elements are defined by

$$\mathrm{pf}(d_m, i)_n = \psi_i(n+m), \qquad \mathrm{pf}(d_m, d_k)_n = \mathrm{pf}(i, j)_n = 0, \qquad i, j = 1, 2, \ldots, N.$$

Then we obtain the following differential formula from expressions (9) and (10):

$$\begin{aligned} f_{nx} = {} & \sum_{j=1}^{K} \dot{\beta}_j(x) \mathrm{pf}(d_0, \ldots, d_{N-1}, N, \ldots, \hat{j}, \ldots, 1, c_j)_n \\ & + \mathrm{pf}(d_{-1}, d_0, d_2, \ldots, d_{N-1}, N, \ldots, 2, 1)_n \\ & - \mathrm{pf}(d_{-2}, d_1, \ldots, d_{N-1}, N, \ldots, 2, 1)_n, \end{aligned} \tag{12}$$

where the dot denotes the derivative of the function $\beta_j(x)$ with respect to x, and new Pfaffian entries are defined by

$$\mathrm{pf}(d_m, c_i)_n = \varphi_{i2}(n+m), \qquad \mathrm{pf}(c_i, c_j)_n = \mathrm{pf}(c_i, j)_n = 0, \qquad i, j = 1, 2, \ldots, N.$$

In addition, we have some other differential and difference formulae

$$\begin{aligned} f_{nz} &= -\mathrm{pf}(d_{-1}, d_1, \ldots, d_{N-1}, N, \ldots, 2, 1), \\ f_{ny} &= \mathrm{pf}(d_0, d_1, \ldots, d_{N-2}, d_N, N, \ldots, 2, 1), \\ f_{n+1} &= \mathrm{pf}(d_1, d_2, \ldots, d_N, N, \ldots, 2, 1). \end{aligned} \tag{13}$$

From expressions (12) and (13), we find that f_n's no longer satisfy equations (3) and (4). Hence we need to introduce new functions $g_{j,n}, h_{j,n}$, which are expressed in Pfaffian forms as

$$g_{j,n} \triangleq g_j(n) = \sqrt{2\dot{\beta}_j(x)}\mathrm{pf}(d_{-1}, d_0, \ldots, d_{N-1}, N, \ldots, 1, c_j)_n, \tag{14}$$

$$h_{j,n} \triangleq h_j(n) = \sqrt{2\dot{\beta}_j(x)}\mathrm{pf}(d_1, d_2, \ldots, d_{N-1}, N, \ldots, \hat{j}, \ldots, 1)_n, \tag{15}$$

where $j = 1, 2, \ldots, K$. We can show that the f_n, $g_{j,n}$ and $h_{j,n}$ so defined satisfy the bilinear equations

$$(D_y D_z - 2\,\mathrm{e}^{D_n} + 2) f_n \cdot f_n = 0, \tag{16}$$

$$(D_y D_x - 2D_z\,\mathrm{e}^{D_n}) f_n \cdot f_n = -\sum_{j=1}^{K} \mathrm{e}^{D_n} g_{j,n} \cdot h_{j,n}, \tag{17}$$

$$(D_y + \mathrm{e}^{-D_n}) f_n \cdot g_{j,n} = 0, \qquad j = 1, 2, \ldots, K, \tag{18}$$

$$(D_y + \mathrm{e}^{-D_n}) h_{j,n} \cdot f_n = 0, \qquad j = 1, 2, \ldots, K. \tag{19}$$

In the following, we prove that f_n, $g_{j,n}$ and $h_{j,n}$ are solutions of equations (16)–(19). Firstly, from the expression of f_n and section 1, we can easily find that equation (16) holds. On the other hand, for simplicity of calculation, we set

$$\begin{aligned} \tilde{g}_{j,n} &= \mathrm{pf}(d_{-1}, d_0, \ldots, d_{N-1}, N, \ldots, 1, c_j), \\ \tilde{h}_{j,n} &= \mathrm{pf}(d_1, d_2, \ldots, d_{N-1}, N, \ldots, \hat{j}, \ldots, 1). \end{aligned}$$

Then we get the formulae

$$\begin{aligned} \tilde{g}_{j,n+1} &= \mathrm{pf}(d_0, d_1, \ldots, d_N, N, \ldots, 1, c_j), \\ \tilde{g}_{j,ny} &= \mathrm{pf}(d_{-1}, \ldots, d_{N-2}, d_N, N, \ldots, 1, c_j), \end{aligned} \tag{20}$$

$$\begin{aligned} \tilde{h}_{j,n-1} &= \mathrm{pf}(d_0, d_1, \ldots, d_{N-2}, N, \ldots, \hat{j}, \ldots, 1), \\ \tilde{h}_{j,ny} &= \mathrm{pf}(d_1, \ldots, d_{N-2}, d_N, N, \ldots, \hat{j}, \ldots, 1). \end{aligned} \tag{21}$$

Substituting expressions (12), (13) and (20), (21) into equation (17), we get the sum of the determinant identities

$$\begin{aligned} 0 = \sum_{j=1}^{K} \dot{\beta}_j(x)[&\mathrm{pf}(d_0, \ldots, d_{N-1}, \star)\mathrm{pf}(d_0, \ldots, d_{N-2}, d_N, N, \ldots, \hat{j}, \ldots, 1, c_j) \\ &- \mathrm{pf}(d_0, \ldots, d_{N-2}, d_N, \star)\mathrm{pf}(d_0, \ldots, d_{N-1}, N, \ldots, \hat{j}, \ldots, 1, c_j) \\ &+ \mathrm{pf}(d_0, \ldots, d_N, \star, c_j)\mathrm{pf}(d_0, \ldots, d_{N-2}, N, \ldots, \hat{j}, \ldots, 1)], \end{aligned}$$

where $\star$ denotes $\{N, \ldots, 2, 1\}$. The above result indicates that equation (17) holds. Similarly, substitution of (13) and (20) into equation (18) yields the Maya diagram of determinants

$$\begin{aligned}&\mathrm{pf}(d_{-1}, d_0, \ldots, d_{N-2}, \star)\mathrm{pf}(d_0, \ldots, d_N, \star, c_j)\\&\quad+\mathrm{pf}(d_0, \ldots, d_{N-2}, d_N, \star)\mathrm{pf}(d_{-1}, \ldots, d_{N-1}, \star, c_j)\\&\quad-\mathrm{pf}(d_0, d_1, \ldots, d_{N-1}, \star)\mathrm{pf}(d_{-1}, \ldots, d_{N-2}, d_N, \star, c_j) = 0,\end{aligned}$$

which indicates equation (18) holds. In an analogous way, we can prove f_n and $h_{j,n}$ satisfy equation (19). Therefore f_n, $g_{j,n}$ and $h_{j,n}$ in (8) and (14), (15) are the Casoratian determinant solution of equations (16)–(19), which are just the two-dimensional Leznov lattice equations with self-consistent sources (LeznovESCS) in the bilinear form.

If we apply the dependent variable transformations

$$\theta(n) = \frac{f(n+1)f(n-1)}{f(n)^2}, \qquad p(n) = \frac{1}{2}\frac{D_xD_y f(n)\cdot f(n)}{f(n+1)f(n-1)},$$
$$v_j(n) = \frac{g_j(n)}{f(n)}, \qquad w_j(n) = \frac{h_j(n)}{f(n)}$$

the bilinear LeznovESCS (16)–(19) is transformed into the nonlinear equations

$$\frac{\partial^2}{\partial x\partial y}\ln\theta(n) = \theta(n+1)p(n+1) - 2\theta(n)p(n) + \theta(n-1)p(n-1), \tag{22}$$

$$\frac{\partial p(n)}{\partial y} = \theta(n+1) - \theta(n-1) - \frac{1}{2}\sum_{j=1}^{K}(v_j(n+1)w_j(n-1))_y, \tag{23}$$

$$\frac{\partial v_j(n)}{\partial y} = \theta(n)v_j(n+1), \qquad j = 1, 2, \cdots, K, \tag{24}$$

$$\frac{\partial w_j(n)}{\partial y} = -\theta(n)w_j(n-1), \qquad j = 1, 2, \cdots, K. \tag{25}$$

When $g_{j,n}$ and $h_{j,n}$ in (16)–(19) are selected to be zero, equations (16)–(19) are reduced to the bilinear two-dimensional Leznov lattice equations (3) and (4). Accordingly, equations (22)–(25) are reduced to the two-dimensional Leznov lattice equations (1), (2).

3. Grammian determinant solution of the LeznovESCS (16)–(19)

In section 2, we have constructed the LeznovESCS and obtained the Casoratian determinant solution of the LeznovESCS through the source generation procedure. We know from [26, 29] that we can also get the Grammian determinant solution of the LeznovESCS through the same procedure. Following the procedure, we first give a kind of Grammian determinant solution of equations (3) and (4)

$$\tau_n \triangleq \tau(n) = |c_{ij} + (-1)^n \int^y \varphi_i(n)\phi_j(-n)\,\mathrm{d}y|_{1\leqslant i,j\leqslant N}, \tag{26}$$

where c_{ij} is a constant. Here each function $\varphi_i(n)$ still satisfies relation (6) and each function $\phi_i(-n)$ satisfies the following relations:

$$\frac{\partial\phi_i(-n)}{\partial y} = \phi_i(-n+1), \qquad \frac{\partial\phi_i(-n)}{\partial z} = -\phi_i(-n-1), \qquad \frac{\partial\phi_i(-n)}{\partial x} = \phi_i(-n-2). \tag{27}$$

It can be proved that the τ_n in (26) satisfies equations (3) and (4) through determinant identities. Now we generalize this τ_n into the function

$$F(n) = |c_{ij}(x) + (-1)^n \int^y \varphi_i(n)\phi_j(-n)\mathrm{d}y|_{1\leqslant i,j\leqslant N}$$
$$= \mathrm{pf}_1(1,2,\ldots,N,N^*,\ldots,2^*,1^*)_n \triangleq F_n, \tag{28}$$

where Pfaffian elements are defined by

$$\mathrm{pf}_1(i,j^*)_n = c_{ij}(x) + (-1)^n \int^y \varphi_i(n)\phi_j(-n)\,\mathrm{d}y,$$
$$\mathrm{pf}_1(i,j)_n = \mathrm{pf}_1(i^*,j^*)_n = 0, \qquad i,j = 1,2,\ldots,N,$$

and $c_{ij}(x)$ satisfies

$$c_{ij}(x) \equiv \begin{cases} \beta_i(x), & i = j \quad \text{and} \quad 1 \leqslant i \leqslant K \leqslant N, K, N \in Z^+, \\ c_{ij}, & i \neq j \quad \text{and} \quad 1 \leqslant i, j \leqslant N. \end{cases}$$

Then we find that F_n does not satisfy equations (3) and (4) again. So we introduce other new functions expressed in Pfaffian form:

$$G_{j,n} \triangleq G_j(n) = \sqrt{2\dot{\beta}_j(x)}\mathrm{pf}_1(d^*_{-1},1,\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n, \tag{29}$$

$$H_{j,n} \triangleq H_j(n) = \sqrt{2\dot{\beta}_j(x)}\mathrm{pf}_1(d_{-1},1,\ldots,\hat{j},\ldots,N,N^*,\ldots,1^*)_n, \tag{30}$$

where $j = 1,2,\ldots,K$, and the new Pfaffian entries are defined by

$$\mathrm{pf}_1(d^*_m,i)_n = \varphi_i(n+m), \qquad \mathrm{pf}_1(d_m,j^*)_n = (-1)^{n-m}\phi_i(-n+m),$$

$$\mathrm{pf}_1(d_m,d^*_l)_n = \mathrm{pf}_1(d_m,d_l)_n = \mathrm{pf}_1(d^*_m,d^*_l)_n = \mathrm{pf}_1(d_m,i)_n = \mathrm{pf}_1(d^*_m,j^*)_n = 0.$$

We can see that F_n, $G_{j,n}$ and $H_{j,n}$ so defined satisfy bilinear equations (16)–(19), i.e.,

$$(D_yD_z - 2\,\mathrm{e}^{D_n} + 2)F_n \cdot F_n = 0, \tag{31}$$

$$(D_yD_x - 2D_z\,\mathrm{e}^{D_n})F_n \cdot F_n = -\sum_{j=1}^{K} \mathrm{e}^{D_n} G_{j,n} \cdot H_{j,n}, \tag{32}$$

$$(D_y + \mathrm{e}^{-D_n})F_n \cdot G_{j,n} = 0, \qquad j = 1,2,\ldots,K, \tag{33}$$

$$(D_y + \mathrm{e}^{-D_n})H_{j,n} \cdot F_n = 0, \qquad j = 1,2,\ldots,K. \tag{34}$$

In fact, it is obvious that F_n still satisfy equation (31). On the other hand, from expression (28), we have the differential formulae

$$\frac{\partial f_n}{\partial x} = \sum_{j=1}^{K} \dot{\beta}_j(x)\mathrm{pf}_1(1,2,\ldots,\hat{j},\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n$$
$$+ \mathrm{pf}_1(d_{-1},d^*_{-2},\bullet)_n + \mathrm{pf}_1(d_{-2},d^*_{-1},\bullet)_n, \tag{35}$$

$$\frac{\partial^2}{\partial x\partial y} f_n = \sum_{j=1}^{K} \dot{\beta}_j(x)\mathrm{pf}_1(d_0,d^*_0,1,2,\ldots,\hat{j},\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n$$
$$+ \mathrm{pf}_1(d_0,d^*_{-2},\bullet)_n + 2\mathrm{pf}_1(d_{-1},d^*_{-1},\bullet)_n + \mathrm{pf}_1(d_{-2},d^*_0,\bullet)_n$$
$$+ \mathrm{pf}_1(d_{-2},d^*_{-1},d_0,d^*_0,\bullet)_n + \mathrm{pf}_1(d_{-1},d^*_{-2},d_0,d^*_0,\bullet)_n. \tag{36}$$

Substituting (29), (30) and (35), (36) into equation (32) yields the sum of Jacobi determinant identities

$$\sum_{j=1}^{K}\dot{\beta}_j(x)[\mathrm{pf}_1(d_0,d_0^*,1,2,\ldots,\hat{j},\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n\mathrm{pf}_1(\bullet)_n$$
$$-\mathrm{pf}_1(1,2,\ldots,\hat{j},\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n\mathrm{pf}_1(d_0,d_0^*,\bullet)_n$$
$$+\mathrm{pf}_1(d_0^*,1,\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n$$
$$\times\,\mathrm{pf}_1(d_0,1,\ldots,\hat{j},\ldots,N,N^*,\ldots,1^*)_n]=0.$$

Similarly, we can prove that equation (33) can be transformed into another Jacobi determinant identity

$$\mathrm{pf}_1(d_0,d_0^*,\bullet)_n\mathrm{pf}_1(d_{-1}^*,1,\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n$$
$$-\mathrm{pf}_1(\bullet)_n\mathrm{pf}_1(d_{-1}^*,d_0,d_0^*,1,\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n$$
$$-\mathrm{pf}_1(d_0,d_{-1}^*,\bullet)_n\mathrm{pf}_1(d_0^*,1,\ldots,N,N^*,\ldots,\hat{j}^*,\ldots,1^*)_n=0,$$

where $\bullet$ denotes $\{1,2,\ldots,N,N^*,\ldots,1^*\}$. The above results indicate that equations (32) and (33) hold. Much in the same way, equation (34) holds. Hence F_n, $G_{j,n}$ and $H_{j,n}$ in (28)–(30) are the Grammian determinant solutions of the bilinear LeznovESCS (16)–(19).

4. Bilinear Bäcklund transformation of equations (16)–(19)

In this section, we present a bilinear Bäcklund transformation for the LeznovESCS (16)–(19). The results obtained are:

Proposition 1. *The bilinear system (16)–(19) has the bilinear Bäcklund transformation:*

$$(D_y+\lambda\,\mathrm{e}^{-D_n}+\theta)f_n\cdot f_n'=0, \tag{37}$$
$$(D_y+\lambda\,\mathrm{e}^{-D_n}+\theta)g_{j,n}\cdot g_{j,n}'=0,\qquad j=1,2,\ldots,K, \tag{38}$$
$$(D_y+\lambda\,\mathrm{e}^{-D_n}+\theta)h_{j,n}\cdot h_{j,n}'=0,\qquad j=1,2,\ldots,K, \tag{39}$$
$$\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot h_{j,n}'=\lambda\,\mathrm{e}^{-\frac{1}{2}D_n}f_n\cdot h_{j,n}'+\mu_j\,\mathrm{e}^{-\frac{1}{2}D_n}h_{j,n}\cdot f_n',\qquad j=1,2,\ldots,K, \tag{40}$$
$$\mathrm{e}^{\frac{1}{2}D_n}g_{j,n}\cdot f_n'=\lambda\,\mathrm{e}^{-\frac{1}{2}D_n}g_{j,n}\cdot f_n'+\mu_j\,\mathrm{e}^{-\frac{1}{2}D_n}f_n\cdot g_{j,n}',\qquad j=1,2,\ldots,K, \tag{41}$$
$$(D_z\,\mathrm{e}^{-\frac{1}{2}D_n}-\lambda^{-1}\,\mathrm{e}^{\frac{1}{2}D_n}+\gamma\,\mathrm{e}^{-\frac{1}{2}D_n})f_n\cdot f_n'=0, \tag{42}$$
$$(\lambda D_x\,\mathrm{e}^{-\frac{1}{2}D_n}-D_z\,\mathrm{e}^{\frac{1}{2}D_n}-\gamma\,\mathrm{e}^{\frac{1}{2}D_n}+\nu\,\mathrm{e}^{-\frac{1}{2}D_n})f_n\cdot f_n'=\sum_{j=1}^{K}\frac{\lambda}{2\mu_j}\,\mathrm{e}^{\frac{1}{2}D_n}g_{j,n}\cdot h_{j,n}', \tag{43}$$

where $\lambda,\theta,\gamma,\nu$, and μ_j are arbitrary constants.

Proof. Let $(f_n,g_{j,n},h_{j,n})$ be a solution of equations (16)–(19) and $(f_n',g_{j,n}',h_{j,n}')$ satisfies relations (37)–(43). What we need to prove is that $(f_n',g_{j,n}',h_{j,n}')$ is also a solution of equations (16)–(19). According to [25], we know that f_n', $g_{j,n}'$ and $h_{j,n}'$ satisfy equations (16) and (18), (19). Hence we only need to prove that equation (17) holds. In fact, using the bilinear operator identities (A.1)–(A.6) and relations (37)–(43), we derive

$$P\equiv\left\{(D_yD_x-2D_z\,\mathrm{e}^{D_n})f_n\cdot f_n+\sum_{j=1}^{K}\mathrm{e}^{D_n}g_{j,n}\cdot h_{j,n}\right\}(f_n')^2-f_n^2\left\{(D_yD_x-2D_z\,\mathrm{e}^{D_n})f_n'\cdot f_n'\right.$$
$$\left.+\sum_{j=1}^{K}\mathrm{e}^{D_n}g_{j,n}'\cdot h_{j,n}'\right\}=2D_x(D_yf_n\cdot f_n')\cdot f_nf_n'-4\sinh\frac{D_n}{2}\left[\left(D_z\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n'\right)\right.$$

$$
\cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot f_n'\big) - \big(\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n'\big) \cdot \big(D_z\, \mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot f_n'\big)\big] + \sum_{j=1}^{K} \big[\mathrm{e}^{\frac{1}{2}D_n} \big(\mathrm{e}^{\frac{1}{2}D_n} g_{j,n} \cdot f_n'\big)
$$

$$
\cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} h_{j,n} \cdot f_n'\big) - \mathrm{e}^{-\frac{1}{2}D_n} \big(\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot h_{j,n}'\big) \cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot g_{j,n}'\big)\big]
$$

$$
= -2\lambda D_x \big(\mathrm{e}^{-D_n} f_n \cdot f_n'\big) \cdot f_n f_n' - 4 \sinh \frac{D_n}{2} \big(D_z\, \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n'\big) \cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot f_n'\big)
$$

$$
- 4\gamma \sinh \frac{D_n}{2} \big(\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n'\big) \cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot f_n'\big) - \sum_{j=1}^{K} \frac{\lambda}{\mu_j} \big[\mathrm{e}^{\frac{1}{2}D_n} \big(\mathrm{e}^{\frac{1}{2}D_n} g_{j,n} \cdot f_n'\big)
$$

$$
\cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot h_{j,n}'\big) - \mathrm{e}^{\frac{1}{2}D_n} \big(\mathrm{e}^{-\frac{1}{2}D_n} g_{j,n} \cdot f_n'\big) \cdot \big(\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot h_{j,n}'\big)\big]
$$

$$
= -4\nu \sinh \frac{D_n}{2} \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot f_n'\big) \cdot \big(\mathrm{e}^{-\frac{1}{2}D_n} f_n \cdot f_n'\big) \equiv 0. \tag{44}
$$

The above result indicates that f_n', $g_{j,n}'$ and $h_{j,n}'$ satisfy equation (17). Therefore we have completed the proof. □

5. Conclusion and discussions

In this paper, we constructed and solved the two-dimensional LeznovESCS through the source generation procedure, starting from the Casoratian determinant and Grammian determinant solutions of the Leznov lattice equation, respectively. As a result, we have obtained the Casoratian determinant solution and the Grammian determinant solution of the LeznovESCS, similar to the fact that the Leznov lattice equation has two kinds of determinant solutions. When the LeznovESCS possesses $K\,(K \geqslant 1)$ pairs of sources, we can obtain N-order $(N \geqslant K)$ Casoratian determinant and N-order Grammian determinant solutions of the system. If we set each $\beta_j(x)$ be a constant, the sources $g_{j,n}, h_{j,n}, G_{j,n}$ and $H_{j,n}$ all become zero. Then the LeznovESCS is reduced to the Leznov lattice equation. Accordingly, f_n and F_n are reduced to the Casoratian determinant and Grammian determinant solutions of the Leznov lattice equation, respectively. In this case, solutions of the LeznovESCS are a kind of generalization of solutions to the Leznov lattice equation. It is noted that the Leznov lattice equation can be generalized into more extended forms. For example, we can define the function $\psi_i(n)$ in (9) in the following new form:

$$
\psi_i(n) = \sum_{k=1}^{K} C_{ik}(x) \varphi_{ik}(n),
$$

where each $\varphi_{ik}(n)$ satisfies relation (6), and each $C_{ik}(x)$ is some arbitrary function. Similarly, for the Grammian determinant F_n in (28), we can also select the arbitrary function $c_{ij}(x)$ in the following form:

$$
c_{ij}(x) = c_{ij}' + \sum_{k=1}^{K} \int^x \beta_{ik}(x) \gamma_{jk}(x)\, \mathrm{d}x, \qquad c_{ij}' = \text{constant}, \qquad 1 \leqslant i, j \leqslant N
$$

where $\beta_{ik}(x)$ and $\gamma_{jk}(x)$ are arbitrary functions.

Acknowledgments

The authors would like to express their sincere thanks to the referees for their valuable advice. This work was partially supported by the National Natural Science Foundation of China (grant no 10471139), CAS President grant, the knowledge innovation program of the

Institute of Computational Math., AMSS and Hong Kong Research Grant Council grant no HKBU2016/05P.

Appendix. Hirota's bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a_n, b_n, c_n, a'_n, b'_n and c'_n.

$$(D_y D_x a_n \cdot a_n) b_n^2 - a_n^2 (D_y D_x b_n \cdot b_n) = 2D_x(D_y a_n \cdot b_n) \cdot a_n b_n, \tag{A.1}$$

$$\begin{aligned}(D_z \mathrm{e}^{D_n} a_n \cdot a_n) b_n^2 - a_n^2 (D_z \mathrm{e}^{D_n} b_n \cdot b_n) &= 2\sinh\left(\frac{D_n}{2}\right)\left[\left(D_z \mathrm{e}^{\frac{D_n}{2}} a_n \cdot b_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b_n\right)\right. \\ &\quad \left. - \left(\mathrm{e}^{\frac{D_n}{2}} a_n \cdot b_n\right) \cdot \left(D_z \mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b_n\right)\right],\end{aligned} \tag{A.2}$$

$$D_x(\mathrm{e}^{-D_n} a_n \cdot b_n) \cdot a_n b_n = -2\sinh\left(\frac{D_n}{2}\right)\left(D_x \mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b_n\right), \tag{A.3}$$

$$\begin{aligned}(\mathrm{e}^{D_n} b_n \cdot c_n)(a'_n)^2 - a_n^2(\mathrm{e}^{D_n} b'_n \cdot c'_n) &= \mathrm{e}^{\frac{D_n}{2}}\left(\mathrm{e}^{\frac{D_n}{2}} b_n \cdot a'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} c_n \cdot a'_n\right) \\ &\quad - \mathrm{e}^{-\frac{D_n}{2}}\left(\mathrm{e}^{\frac{D_n}{2}} a_n \cdot c'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b'_n\right),\end{aligned} \tag{A.4}$$

$$\begin{aligned}(D_z \mathrm{e}^{D_n} b_n \cdot c_n)(a'_n)^2 - a_n^2(D_z \mathrm{e}^{D_n} b'_n \cdot c'_n) &= D_z \mathrm{e}^{\frac{D_n}{2}}\left(\mathrm{e}^{\frac{D_n}{2}} b_n \cdot a'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} c_n \cdot a'_n\right) \\ &\quad + D_z \mathrm{e}^{-\frac{D_n}{2}}\left(\mathrm{e}^{\frac{D_n}{2}} a_n \cdot c'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b'_n\right),\end{aligned} \tag{A.5}$$

$$\begin{aligned}2\sinh\left(\frac{D_n}{2}\right)\left(\mathrm{e}^{\frac{D_n}{2}} b_n \cdot b'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} a_n \cdot a'_n\right) &= \mathrm{e}^{\frac{D_n}{2}}\left(\mathrm{e}^{\frac{D_n}{2}} b_n \cdot a'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} a_n \cdot b'_n\right) \\ &\quad - \mathrm{e}^{-\frac{D_n}{2}}\left(\mathrm{e}^{\frac{D_n}{2}} a_n \cdot b'_n\right) \cdot \left(\mathrm{e}^{-\frac{D_n}{2}} b_n \cdot a'_n\right).\end{aligned} \tag{A.6}$$

References

[1] Antonowicz M and Rauch-Wojciechowski S 1993 Soliton hierarchies with sources and Lax representation for restricted flows *Inverse Problems* **9** 201–5

[2] Mel'nikov V K 1988 Exact solutions of the Korteweg–de Vries equation with a self-consistent source *Phys. Lett.* A **128** 488–92

[3] Mel'nikov V K 1989 Capture and confinement of solitons in nonlinear integrable systems *Commun. Math. Phys.* **120** 451–68

[4] Zeng Y B, Ma W X and Lin R L 2000 Integration of the soliton hierarchy with self-consistent sources *J. Math. Phys.* **41** 5453–89

[5] Zeng Y B, Ma W X and Shao Y J 2001 Two binary Darboux transformations for the KdV hierarchy with self-consistent sources *J. Math. Phys.* **42** 2113–28

[6] Leon J and Latifi A 1990 Solution of an initial-boundary value problem for coupled nonlinear waves *J. Phys. A: Math. Gen.* **23** 1385–403

[7] Mel'nikov V K 1987 A direct method for deriving a multi-soliton solution for the problem of interaction of waves on the x, y plane *Commun. Math. Phys.* **112** 639–52

[8] Mel'nikov V K 1990 Integration of the Korteweg–de Vries equation with a source *Inverse Problems* **6** 233–46

[9] Lin R L, Zeng Y B and Ma W X 2001 Solving the KdV hierarchy with self-consistent sources by inverse scattering method *Physica* A **291** 287–98

[10] Urasboev G U and Khasanov A B 2001 Integration of the Korteweg–de Vries equation with a self-consistent source in the presence of 'steplike' initial data *Theor. Math. Phys.* **129** 1341–56

[11] Ye S and Zeng Y B 2002 Integration of the modified Korteweg–de Vries hierarchy with an integral type of source *J. Phys. A: Math. Gen.* **35** L283–91

[12] Ma W X 2003 Soliton, positon and negaton solutions to a Schrödinger self-consistent source equation *J. Phys. Soc. Japan* **72** 3017–9

[13] Zeng Y B 1994 New factorization of the Kaup-Newell hierarchy, *Physica* D **73** 171–88
[14] Zeng Y B, Shao Y J and Ma W X 2002 Integral-type Darboux transformation for the mKdV hierarchy with self-consistent sources *Commun. Theor. Phys.* (Beijing) **38**(6) 641–8
[15] Xiao T and Zeng Y B 2004 Generalized Darboux transformations for the KP equation with self-consistent sources *J. Phys. A: Math. Gen.* **37** 7143–62
[16] Xiao T and Zeng Y B 2005 A new constrained mKP hierarchy and the generalized Darboux transformation for the mKP equation with self-consistent sources *Physica* A **353** 38–60
[17] Hase Y, Hirota R, Ohta Y and Satsuma J 1989 Soliton solutions of the Mel'nikov equations *J. Phys. Soc. Japan* **58** 2713–20
[18] Hu X B 1991 Nonlinear superposition formula of the KdV equation with a source *J. Phys. A: Math. Gen.* **24** 5489–97
[19] Hu X B 1996 The higher-order KdV equation with a source and nonlinear superposition formula *Chaos Solitons Fractals* **7** 211–5
[20] Matsuno Y 1990 KP equation with a source and its soliton solutions *J. Phys. A: Math. Gen.* **23** L1235–9
[21] Gegenhasi and Hu X B 2006 On a integrable differential-difference equation with a source *J. Nonlinear Math. Phys.* **13**(2) 183–92
[22] Zhang D J 2002 The N-soliton solutions for the modified KdV equation with self-consistent sources *J. Phys. Soc. Japan* **71** 2649–56
[23] Deng S F, Chen D Y and Zhang D J 2003 The multisoliton solutions of the KP equation with self-consistent sources *J. Phys. Soc. Japan* **72** 2184–92
[24] Gegenhasi and Hu X B Integrability of a differential-difference KP equation with self-consistent sources *Math. Comput. Simul.* doi:10.1016/j.matcom.2006.10.034
[25] Wang H Y, Hu X B and Gegenhasi 2007 2D Toda lattice equation with self-consistent sources: Casoratian type solution, bilinear Bäcklund transformation and Lax pair *J. Comp. Appl. Math.* doi:10.1016/j.cam.2005.08.052
[26] Hu X B and Wang H Y 2006 Construction of dKP and BKP equations with self-consistent sources *Inverse Problems* **22** 1903–20
[27] Wang H Y 2006 Interability of the semi-discrete Toda equation with self-consistent sources *J. Math. Anal. Appl.* at press
[28] Leznov A N 2000 Graded Lie algebras, representation theory, integrable mappings and integrable systems *Theor. Math. Phys.* **122** 211–28
[29] Hirota R 2004 *The Direct Method in Soliton Theory* ed A Nagai, J Nimmo and C Gilson (Cambridge: Cambridge University Press)
[30] Hu X B and Tam H W 2000 Application of Hirota's bilinear formalism to a two-dimensional lattice by Leznov *Phys. Lett.* A **276** 65–72
[31] Zhao J X, Hu X B and Tam H W 2005 Application of pfaffianization procedure to the two-dimensional Leznov lattice, *Theor. Math. Phys.* **144** 1288–95